

Exact Analytic Solution of Fokker-Planck Equation for Propagation of Particles Scattered Isotropically

M.A. Malkov

¹*University of California San Diego, La Jolla, CA 92093*

An analytic solution for a Fokker-Planck equation that describes propagation of energetic particles through a scattering medium is obtained. The solution is found in terms of an infinite series of mixed moments of particle distribution, $\langle \mu^j x^k \rangle$, where x is the particle displacement and μ is its normalized velocity projection on the travel direction. The spatial dispersion of a particle cloud released at $t = 0$ is described by a simple formula ($j = 0, k = 2$) obtained by G.I. Taylor (1920) in his classical study of *random walk*: $\langle x^2 \rangle = \langle x^2 \rangle_0 + t/3 + [\exp(-2t) - 1]/6$, shown here in time/length units equal to the particle collision time/mean free path, with $\langle x^2 \rangle_0$ being an initial dispersion of the cloud. This formula distills a transition from ballistic (rectilinear) propagation phase, $\langle x^2 \rangle - \langle x^2 \rangle_0 \approx t^2/3$ to a time-asymptotic, diffusive phase, $\langle x^2 \rangle - \langle x^2 \rangle_0 \approx t/3$. The present paper provides all the higher moments by a recurrent formula. The full set of moments is equivalent to the full solution of the Fokker-Planck equation, expressed in form of an infinite series in moments $\langle \mu^j x^k \rangle$. An explicit, easy-to-use approximation for a point source spreading of a pitch-angle averaged distribution $f_0(x, t)$ (starting from $f_0(x, 0) = \delta(x)$), is presented and verified by a numerical integration of the Fokker-Planck equation.

I. INTRODUCTION

Propagation of energetic particles, that we will call cosmic rays (or CRs for short), particularly through magnetized turbulent media, has been actively researched in astrophysical community for more than half a century. The time asymptotic solution of this problem is clearly diffusive. After several collisions particles “forget” their starting velocities, thus entering a random walk propagation regime. Typically, they propagate along an averaged magnetic field direction. In many important cases, however, there is not enough time or room for even a few collisions. In such systems, an early-time propagation is not so random as particles still “remember” their starting velocities and positions.

At times much shorter than the collision time, t_c , most particles propagate with their initial velocity projections on the field direction, if it is present. This regime is called ballistic, or rectilinear propagation. The question then is what happens next, namely at $t \sim t_c$ but before the onset of diffusion at $t \gg t_c$? What exactly is the value of $t/t_c \gg 1$, when it is safe to switch to the simple diffusive description? In other words, what is the extent of a transdiffusive phase when neither ballistic nor diffusive model applies? These are the central questions we address below using a Fokker-Planck transport equation and its exact analytic solution.

A review of early results on CR propagation is contained in Ref.[12], while Ref.[16] covers more recent studies. The Fokker-Planck model is broadly applied by the CR and heliophysics communities to a wide range of transport processes under small stochastic changes in particle velocity direction. This formulation is relevant to such astrophysical environments as the solar wind, through which solar energetic particles and cosmic rays propagate to the Earth, and the interstellar medium, through which CRs propagate from more distant sources. Another example is the propagation of ultra-high-energy CRs from extragalactic sources. A transition from ballistic to diffusive transport regime, while being challenging for the theory, is key to understanding the nature of such sources. Since the particle mean free path usually grows with energy, a significant part of their spectrum falls into a transient category where neither ballistic nor diffusive approximation applies. In fact, these are the particles that in many cases carry the most crucial information about the source. Indeed, the low-energy, diffusively propagating particles either do not reach the observer or merge into a featureless isotropic background. The highest energy particles, on the contrary, propagate ballistically. They could point back to their sources, but they are rare and do not provide statistics. On average, just one CR particle with energy $> 10^{20}$ eV is expected to arrive per century per square kilometer and only a handful of such events have been registered over decades of observation. The number of particles rapidly increases towards lower energies, but their arrival directions are scrambled, even if they come from the same point source. It is, therefore, important to know how significant was the scrambling effect as a function of energy, during their travel to the Earth.

II. FORMULATION OF THE PROBLEM IN ITS HISTORICAL CONTEXT

The solution of the Fokker-Planck equation obtained below can be applied to both magnetized and unmagnetized media, where particles (or other entities such as waves) are randomly scattered at small angles to a local magnetic field direction or any other axis of symmetry along which they propagate. The magnetic field, however, conveniently justifies

the one-dimensional reduction of the more realistic three-dimensional problem, as particles are bound to the field line. After averaging out their gyromotion (usually unimportant) the particle phase space becomes two-dimensional. The transport equation for their distribution function f reduces then to one-dimensional spatial transport constrained by angular scattering:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} (1 - \mu^2) D \frac{\partial f}{\partial \mu}. \quad (1)$$

Here x is the spatial coordinate along which the particle concentration varies (local field direction or symmetry axis), μ is the cosine of the particle pitch angle to the x -axis, v is the magnitude of particle velocity. It is conserved for particle interaction with quasi-static magnetic turbulence. D is the scattering rate (collision frequency).

A simple yet illuminating example that eq.(1) can handle is an instant release of a small cloud of particles into a scattering medium. For instance, galactic supernova remnants, widely believed to generate CRs with energies up to $\sim 10^{15}$ eV, must accelerate CRs in their shock waves and subsequently release them into the turbulent interstellar medium. Mathematically, the question then is how exactly the pitch-angle averaged particle distribution propagates along a magnetic flux tube that intersects the source. It is highly desirable to reach the simplicity of some of the derivatives of eq.(1), such as a diffusion equation [11]. As emphasized earlier, the latter is inadequate during the ballistic and transdiffusive phases. These may comprise a crucial phase of propagation. For example, during this phase CRs may reach a nearby molecular cloud, making themselves visible by interacting with its dense gas. Another example is the propagation of the solar energetic particles to 1AU. In both cases the particle mean free path (m.f.p.) may be larger or comparable to the characteristic scale of the problem and the diffusion approximation does not apply. Note that if the pressure of released particles is of the order or larger than the magnetic pressure of the ambient medium the particles are self-confined by scattering off self-generated MHD waves. Mathematically, this problem is treated differently [18] from what is suggested below, as the scattering frequency D strongly depends on ∇f , whereas in the present paper $D = D(v)$.

A. Simple limiting cases

Before proceeding with the solution of the Fokker-Planck equation (1), it is useful to characterize its limiting cases of ballistic and diffusive propagation. We deduce them directly from eq.(1), by eliminating angular dynamics.

1. Ballistic propagation regime

In the ballistic regime, that strictly applies to times shorter than the collision time, $t \ll t_c \sim 1/D$, the r.h.s. of the equation can be neglected, and the solution is obtained by integrating along the particle trajectories (Liouville's theorem), $x - \mu vt = \text{const}$ with a conserved pitch angle, $\mu = \text{const}$. The solution is simply $f(x, \mu, t) = f(x - v\mu t, \mu, 0)$.

It is sufficient to consider here a point source with initially isotropic distribution: $f(x, \mu, 0) = \frac{1}{2} \delta(x) \Theta(1 - \mu^2)$, where δ and Θ denote the Dirac's delta and Heaviside unit step functions, respectively. From the above solution for $f(x, \mu, t)$, one obtains the ballistic expansion in form of the second momentum, $\langle x^2 \rangle = v^2 t^2 / 3$ by integrating $\frac{1}{2} f = \frac{1}{2} x^2 \delta(x - v\mu t) \Theta(1 - \mu^2)$ over x and μ . The result describes a free escape with the mean square velocity $v/\sqrt{3}$, while the maximum particle velocity (along x) is v . The form of the pitch angle averaged particle distribution, $f_0(x, t) = (2vt)^{-1} \Theta(1 - x^2/v^2 t^2)$, is characterized by an expanding 'box' of decreasing height. Note that this result is completely at odds with the solution of the so called "telegraph" equation that has been put forward as a viable tool for CR propagation problems over the last 50 years. By contrast, an exact solution of eq.(1) obtained below converges to the above box distribution at $t \ll 1$.

2. Diffusive (hyperdiffusive) propagation regime

The second simple regime is diffusive which sets on after $t \gg t_c \sim 1/D$, and we treat it in a way opposite to the above [11]. The r.h.s. of eq.(1) is now the leading term, thus implying that the particle distribution is close to isotropy, $\partial f / \partial \mu \rightarrow 0$. Working to higher orders in anisotropic corrections $\sim 1/D$, and averaging the equation over μ , one obtains the following equation for $f_0(x, t)$ [19]

$$\frac{\partial f_0}{\partial t} - \kappa_2 \frac{\partial^2 f_0}{\partial x^2} = -\kappa_4 \frac{\partial^4 f_0}{\partial x^4} + \kappa_6 \frac{\partial^6 f_0}{\partial x^6} - \dots, \quad (2)$$

with $\kappa_{2n} \sim 1/D^n$. This particular form of expansion is valid under a constraint of symmetry $D(-\mu) = D(\mu)$. Otherwise, also the odd x -derivatives appear on its r.h.s. This case is not considered here for simplicity. The last equation thus corresponds to an asymptotic expansion of the problem in $1/D$. It is valid only for $t \gg t_c \sim 1/D$, and all the residuals of a short-time, ballistic phase of particle propagation are intentionally eliminated. The r.h.s. of the above equation provides a small hyperdiffusive correction and may be omitted, as the higher spatial derivatives quickly decay because of the smoothing effect from the diffusive term on its l.h.s.

The most serious problem with the diffusive approximation is an infinite propagation speed for a certain group of particles that reach a given distance faster than the maximum speed would allow (causality problem, e.g. [2, 4, 21]). Mathematically, the approximation violates an upper bound $|x| \leq vt$ that immediately follows from eq.(1) for a point source solution. There have been many attempts to overcome this problem, but no adequate *ab initio* description of particle spreading that would cover ballistic and diffusive phases was elaborated. Much popularity received the telegraph equation, mentioned above that has been adopted from the studies of *discontinuous* random walk [10] by Ian Axford [3] back in 1965. Many derivations of telegraph equation from eq.(1) and attempts to justify its application to various cosmic ray transport problems have appeared ever since [8, 14, 15, 21]. However, the solutions of this equation have been shown to be pathological (e.g., singular components incompatible with the parent equation, eq.[1] and nonconservation of the total number of particles, which we discuss below).

Both diffusive and “telegraph” approaches are aimed to extract the particle spatial distribution from eq.(1). The elimination of pitch angle, however, has always been a challenge. While the time-asymptotic diffusive regime is well understood, the idea behind the telegraph equation was to span also the transdiffusive evolution, preferably down to the ballistic phase. Recently [19], the derivation of telegraph equation was shown to be inconsistent with a regular Chapman-Enskog asymptotic expansion (elimination of pitch angle dependence) and other properties of eq.(1) mentioned earlier. The higher order Chapman-Enskog expansion, in turn, is valid only for time-asymptotic regimes, $t \gg t_c$, and should be considered as a correction to diffusive treatment. It has been obtained in [19] in the form of a series of hyperdiffusive terms. As in many other asymptotic expansions, when applied outside of their validity range, higher order terms often make the approximation less accurate which was recently demonstrated in Ref.[15], using the numerical integration of eq.(1). By contrast to the hyperdiffusive Chapman-Enskog expansion, the telegraph equation was intended to cover the crossover phase, $t \sim t_c$. Not surprisingly, it failed to provide adequate fits to full numerical solutions, thus confirming its pathological character.

The telegraph equation can most easily be derived directly from the hyperdiffusive corrections in eq.(2), thus inheriting its validity range, $t \gg t_c$. In this appearance, it constitutes another form of small correction to diffusion. Indeed, considering the two terms on the l.h.s. of eq.(2) as leading (which is required by its derivation!) and converting then the fourth spatial derivative (the leading term on the r.h.s) into a second time-derivative, we write: $f_{0xxxx} \simeq f_{0tt}/\kappa_2^2$. By dropping higher x -derivatives one recovers the telegraph equation

$$\frac{\partial^2 f_0}{\partial t^2} - V^2 \frac{\partial^2 f_0}{\partial x^2} + \tau^{-1} \frac{\partial f_0}{\partial t} = 0. \quad (3)$$

At first glimpse, it indeed captures a ballistic (wave-like) propagation of particle bunches at a speed $V = \sqrt{\kappa_2^3/\kappa_4}$. However, the number density in the bunches decays with time at a rate $\tau^{-1} = \kappa_2^2/\kappa_4$ which is nonphysical. Unlike eqs.(1) and (2), this equation does not conserve the number of particles, $N = \int f_0 dx$, automatically. As a remedy, two different solutions are added together, as the equation is linear and the superposition principle applies. A compound fundamental (Green’s function) solution of eq.(3) is constructed to conserve the total number of particles. Its first component is smooth within the characteristics of eq(3), $|x| < Vt$ and zero otherwise, thus it develops discontinuities at $x = \pm Vt$. The number of particles contained in this solution component grows in time from zero as they start spreading ballistically along the characteristics. The second solution, whose sole purpose is to compensate for the nonphysical multiplication of particles in the first component, needs to be taken in even more singular form, $f_a = \frac{1}{2}N(t)[\delta(x - Vt) + \delta(x + Vt)]$. The number of particles in this component decays as $N = N(0) \exp(-t/2\tau)$. It is clear that this auxiliary component is inconsistent with the original equation (1). Indeed, the particle distribution of the form $\propto \delta(x \pm Vt)$ implies that all particles have the same pitch angle. But the operator of angular scattering on the r.h.s. of eq.(1) would momentarily smear out any sharp pitch-angle distribution. Therefore, the $\delta(x \pm Vt)$ components cannot persist.

We conclude that the telegraph equation is related to the parent eq.(1) only at $t \gg t_c$, when the nonphysical δ -peaks die out and the equation can be deduced from the hyperdiffusive expansion in eq.(2), according to the canonical Chapman-Enskog approach. At earlier times, its solution does not match the actual solution of eq.(1) [15]. Interestingly enough, only at $t \gtrsim 10t_c$ the telegraph solution merges with diffusive, hyperdiffusive and direct numerical solution of the parent equation. Even then, nonphysical peaks stemming from the singular part of the solution remain well pronounced (Fig.5 in the above paper).

Notwithstanding the irrelevance of the telegraph solution to early phases of the particle transport, it is interesting to

note that the singular δ - components do arise in a different context of the telegraph equation (apart from transmission lines) associated with a *discontinuous* random walk. In this process, studied in Ref. [10] and earlier by G.I. Taylor [23], particles are allowed to move only at fixed velocities, positive or negative, say $\pm V$, which naturally results in a $\delta(x \pm Vt)$ particle distribution. Statistically, this is similar to the coin tossing with only two possible outcomes. Under a continuous pitch-angle dependence relevant to the CR propagation, the discontinuous random walk restriction corresponds to the particle distribution, concentrated at $\mu = \pm 1$, thus producing the $\delta(x \pm Vt)$ singular components of the telegraph equation. As we noted, however, the underlying $\delta(\mu \pm 1)$ angular distribution is inconsistent with the regular solutions of Fokker-Planck eq.(1) at $t > 0$. Therefore, the telegraph equation cannot be deduced from this equation except for $t \gg t_c$. Being alternatively derived from the discontinuous random walk process, it is unsuitable for describing the transport of energetic particles since their spectrum in velocity projection on the travel direction (pitch angle) is a fundamentally continuous variable.

It follows that apart from the well established diffusive description of eq.(1), valid only at $t \gg t_c \sim 1/D$, there are no viable analytical tools to address the earlier phases of particle propagation. Therefore, direct solution of the Fokker-Planck equation we tackle below is more than motivated.

III. FOKKER-PLANCK EQUATION AND ITS SOLUTION

Before proceeding to the solution of the Fokker-Planck equation, we briefly discuss possible generalizations and limitations. First, we note that the particle m.f.p. usually depends on its energy. This quantity enters eq.(1) as a parameter, that is $D(E)$, which can be scaled out of the equation. If one is interested in considering a more general case with an energy gain/loss term (at rate $\tau^{-1}(E)$)

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial x} \pm \frac{\partial}{\partial E} \frac{E}{\tau} f = \frac{\partial}{\partial \mu} (1 - \mu^2) D \frac{\partial f}{\partial \mu}, \quad (4)$$

the energy dependence may still be removed by a simple change of variables

$$E' = \int \tau(E) dE / E \mp t, \quad F = Ef / \tau \quad (5)$$

The equation then takes the form of eq.(1)

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial x} = \frac{\partial}{\partial \mu} (1 - \mu^2) D(E', t, \mu) \frac{\partial F}{\partial \mu},$$

thus containing the particle energy, again, only as a parameter. The price to pay is the time dependence of the scattering frequency D . A possible pitch-angle dependence of the scattering rate $D(\mu)$ in eq.(1) may place further limitations on the solution obtained below. In media with magnetic irregularities, $D(\mu)$ derives from a power index of the scattering turbulence, q , if the interaction between particles and turbulence is resonant, e.g. [5, 11, 13, 22, 24]. So, for the power spectrum $P \propto k^{-q}$, where k is the wave number, one obtains $D(\mu) \propto |\mu|^{q-1}$. More complex, anisotropic spectra, such as those derived by [9], result in $D(\mu)$ with a shallow dependence on the pitch angle [7], except for relatively narrow regions near $\mu = 0, \pm 1$. These areas require special consideration as they are strongly affected by particle mirroring and resonance broadening [1] ($\mu \approx 0$) and field aligned propagation [17] ($|\mu| \approx 1$). Below, we assume the $D(\mu) = \text{const}$ scattering coefficient since it is both physically interesting (Goldreich-Shridhar and $q = 1$ cases) and allows for an exact analytic solution of eq.(1). Also, fluctuation spectra with the index $q \approx 1$ have been observed in Monte-Carlo studies of shock-accelerated particles [6]. Returning to the energy dependence of D and the energy loss/acceleration term, we also restrict our consideration below to the simple possible case, $D = D(E)$ and $\tau \rightarrow \infty$. However, using the transformation of variables in eq.(5), our results can be generalized to certain forms of $\tau(E) < \infty$. This generalization needs to be done on a case by case basis, depending on the functional forms of $\tau(E)$ and $D(E)$ which we do not consider here.

We now rewrite eq.(1) using dimensionless time and length units according to the following transformations

$$Dt \rightarrow t, \quad \frac{D}{v} x \rightarrow x$$

Instead of eq.(1) we thus have

$$\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial f}{\partial \mu} \quad (6)$$

This equation contains no explicit parameters. Therefore, we should either solve it exactly or impose restrictions on the initial profile $f(x, \mu, 0)$. In particular, if one is interested in an isotropic approximation that can be treated using eq.(2), ($1/D$ type expansion), the initial distribution not only should be close to isotropy but also it should be spatially broad, $f^{-1} |\partial f / \partial x| \ll 1$. This condition prevents a high anisotropy otherwise produced via the second term on the l.h.s. Hence, the fundamental problem of a point source spreading (Green function, or fundamental solution) can not be treated using the conventional $1/D$ expansion, until f becomes quasi-isotropic, that is broadened to a size $\gtrsim 1$ in x . Therefore, we tackle an exact solution of eq.(6). The only restriction that we impose on the spatial distribution at $t = 0$, which will be maintained in the course of time, is a rapid decay of $f(x)$ at $|x| \rightarrow \infty$. Namely, we require that $x^n f(x) \rightarrow 0$ for $|x| \rightarrow \infty$ and $n \geq 0$. This standard restriction guarantees the existence of all moments. We also assume that f is sufficiently regular at $|\mu| = 1$: $(1 - \mu^2) f \rightarrow 0$ for $|\mu| \rightarrow 1$.

Turning to the solution of eq.(6) we introduce the moments of $f(\mu, x)$ in the form of the following matrix

$$M_{ij}(t) = \langle \mu^i x^j \rangle = \int_{-\infty}^{\infty} dx \int_{-1}^1 \mu^i x^j f d\mu / 2 \quad (7)$$

for any integer $i, j \geq 0$. We will discuss conditions for the equivalence of the moments M and the distribution f when the solution for the matrix M is obtained.

The lowest moment M_{00} is automatically conserved by eq.(6) (as being proportional to the number of particles) and we normalize it to unity

$$M_{00} = \int_{-\infty}^{\infty} dx \int_{-1}^1 f d\mu / 2 = 1.$$

Multiplying eq.(6) by $\mu^i x^j$ and integrating, we obtain the following matrix equation for the moments M_{ij} with $i+j > 0$:

$$\frac{d}{dt} M_{ij} + i(i+1) M_{ij} = j M_{i+1, j-1} + i(i-1) M_{i-2, j} \quad (8)$$

For any $i, j \geq 0$, this equation couples matrix elements from two closest nonadjacent anti-diagonals. One may think that this is a typical infinite system of moment equations that requires closure and truncation. Fortunately, this is not the case. The set of moments $M_{ij}(t)$ can be recursively resolved using eq.(8) to any order $n = i + j$. Indeed, as $M_{00} = 1$, and we can set $M_{ik} = M_{ki} = 0$ for any $i < 0, k \geq 0$, all the elements of the infinite matrix

$$M = \begin{pmatrix} 1 & \langle x \rangle & \langle x^2 \rangle & \langle x^3 \rangle \\ \langle \mu \rangle & \langle \mu x \rangle & \langle \mu x^2 \rangle & \nearrow \\ \langle \mu^2 \rangle & \langle \mu^2 x \rangle & \nearrow & \ddots \\ \langle \mu^3 \rangle & \nearrow & \ddots & \\ \nearrow & \ddots & & \end{pmatrix}$$

on each anti-diagonal can be found, working from left to right as shown by the arrows. The two moments on the uppermost antidiagonal, are easily found from eq.(8) to be $M_{10}(t) = \langle \mu \rangle = \langle \mu \rangle_0 \exp(-2t)$ and $M_{01} = \langle x \rangle = \langle x \rangle_0 + \frac{1}{2} \langle \mu \rangle_0 [1 - \exp(-2t)]$. Higher moments can be obtained inductively. So, in general, from eq.(8) we find

$$M_{ij}(t) = M_{ij}(0) e^{-i(i+1)t} + \int_0^t e^{i(i+1)(t'-t)} [j M_{i+1, j-1}(t') + i(i-1) M_{i-2, j}(t')] dt' \quad (9)$$

As may be seen from the above expressions for M_{01} and M_{10} , all the higher moments can be obtained from eq.(9) in form of series in $t^k e^{-nt}$, where k and n are integral numbers. In particular, the next set of moments is on the third anti-diagonal, M_{20}, M_{11}, M_{02} :

$$M_{20} = \frac{1}{3}, \quad M_{11} = \frac{1}{6} (1 - e^{-2t}), \quad M_{02} = M_{02}(0) + \frac{t}{3} - \frac{1}{6} (1 - e^{-2t}) \quad (10)$$

Being interested in a point source (fundamental) solution we assumed the initial distribution $f(x, \mu, 0)$ to be symmetric in x and isotropic which eliminated the odd moments. Working on such solution the initial spatial width can be set to zero $M_{02}(0) = \langle x^2 \rangle_0 = 0$. Note that $M_{02}(t)$ in eq.(10) coincides with the respective random walk result obtained by G.I. Taylor, which we discussed earlier. However useful for understanding the transition from diffusive to ballistic phase of particle propagation, M_{02} alone does not of course resolve the Fokker-Planck equation. From the mathematical point of view, only a *full* set of moments in eq.(9) provides a complete solution $f(x, \mu, t)$ of eq.(6) given the initial value, $f(x, \mu, 0)$. The latter obviously determines the matrix $M_{ij}(0)$ in eq.(9).

In general, the equivalence between any arbitrary distribution $f(x, \mu, t)$ and its *full* set of moments $M_{ij}(t)$ is not guaranteed automatically, but can be established for eq.(6) with its solution in the form of eq.(9) using Hamburger's theorem, e.g., [20]. The theorem condition is the existence of an upper bound on the moments in form $|M_{ij}| < A n! b^n$ with the constants A and b being independent of n . According to eq.(9), for any fixed $t \gg 1$ the moments $M_{ij}(t)$ grow with $n = i + j$ not faster than $t^{n/2}$. Although for small $t \ll 1$ higher powers of t are present, they also have an upper bound $\sim t^n$. Therefore, the condition for Hamburger theorem is satisfied.

Our focus is on moments that correspond to the isotropic part of particle distribution as it contributes to the particle number density. More specifically, expanding the full particle distribution as

$$f(x, \mu, t) = \sum_{n=0}^{\infty} f_n(x, t) P_n(\mu) \quad (11)$$

we concentrate on f_0 as it constitutes the pitch-angle averaged distribution and is the target of most of the reduction schemes applied to the Fokker-Planck eq.(6). Here P_n are of course the Legendre polynomials. The moments M_{ij} , which represent f_0 are, therefore, M_{0j} . So, we can use a conventional moment-generating function

$$f_\lambda(t) = \int_{-\infty}^{\infty} f_0(x, t) e^{\lambda x} dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{2n}(t) \quad (12)$$

where we have used the notation $M_{2n} \equiv M_{0,2n}$ and omitted the odd moments, as intended earlier. The above expansion will be converted into a standard Fourier transform by setting $\lambda = -ik$. To find its inverse, that is the function $f_0(x, t)$, we will use the inverse Fourier transform

$$f_0(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n}}{(2n)!} M_{2n}(t) \quad (13)$$

To lighten the algebra, we will continue for a while to use λ instead of k . For practical use, we need to simplify the above series in moments M_{2n} . The higher moments (a few of them can be found in Appendix) quickly become unmanageable without computer algebra. In the next section, we will sum up the series in eqs.(12) and (13) by extracting the dominant terms from each moment in the sum, depending on t . To conclude this section, we provide the expression for M_{2n} in eq.(13) through the lower order moment $M_{1,2n-1}$

$$M_{2n}(t) = 2n \int_0^t M_{1,2n-1}(t') dt' \quad (14)$$

The isotropic part of the solution of eq.(6) is thus given by eqs.(13), (14) and (9).

IV. SIMPLIFIED FORMS OF THE SOLUTION

Equations (13) and (14) provide an exact closed form solution of the Fokker-Planck eq.(6). The calculation of the moments M_{2n} , is relatively straightforward. Using eq.(9) and integrating by parts one obtains M_{2n} to any order n in form of polynomials in t and e^{-t} : $\sum_{k,l} C_{kl} t^k e^{-lt}$, with the constant matrix elements C_{kl} that can also be recursively obtained from eq.(9). However, the expressions for M_{2n} grow rapidly in length with n , and some computer algebra is practically required to calculate the series in the Fourier integral in eq.(13). Moreover, as we demonstrated in Sec.II A 1, the early phases of CR propagation are characterized by sharp fronts formed in the profile $f_0(x, t)$. Sharp fronts are dominated by the contributions from $k \gg 1$ in the Fourier integral given by eq.(13). Therefore, the series in n should not be truncated at finite n .

A. Ballistic and transdiffusive phases

We begin summing up the series entering the moment generating function given by eq.(12) for time $t \lesssim 1$ which describes ballistic and transdiffusive phases of CR propagation. Each term $M_{2n}(t)$ of the series is calculated using a three-term expansion in powers of t :

$$f_\lambda = 1 + \frac{\lambda^2 t^2}{2! \cdot 3} \left(1 - \frac{2}{3}t + \frac{t^2}{3} + \dots\right) + \frac{\lambda^4 t^4}{4! \cdot 5} \left(1 - \frac{4}{3}t + \frac{58}{45}t^2 + \dots\right) + \frac{\lambda^6 t^6}{6! \cdot 7} \left(1 - \frac{6}{3}t + \frac{43}{15}t^2 + \dots\right) + \dots \quad (15)$$

The terms in parentheses suggest to introduce the following variable instead of t :

$$t' = t \left(1 - \frac{t}{3}\right).$$

The two leading terms in $t < 1$ at each power of λ can then be written as powers of t'

$$\lambda^n t^n \left(1 - n \frac{t}{3}\right) \approx \lambda^n t'^n$$

and summed up straightforwardly. The remaining terms ($\sim t^2$ in the parentheses in eq.[15]) can also be summed up (see Appendix). To the same order in $t \ll 1$, for arbitrary λt but $\lambda t^2 \ll 1$, we rewrite the above series as follows

$$f_\lambda \approx \frac{1}{\lambda t'} \sinh(\lambda t') e^{\lambda^2 t'^3 / 45} \quad (16)$$

Returning now to the Fourier spectral parameter $k = i\lambda$ and performing the inverse Fourier transform according to eq.(13), from the last expression we obtain

$$f_0(x, t) \approx \frac{1}{4t'} \left[\operatorname{erf}\left(\frac{x + t'}{\Delta}\right) - \operatorname{erf}\left(\frac{x - t'}{\Delta}\right) \right] \quad (17)$$

Here $\Delta = 2t'^{1/2}t'^{3/2}/3\sqrt{5}$, erf is an error function and, again, essential steps of the derivation of the last formula can be found in Appendix. By construction, this simple result constitutes an approximate, pitch angle averaged Green's function for eq.(6), which is also confirmed by the limiting transition $f_0 \rightarrow \delta(x)$, for $t \rightarrow 0$.

According to this simple formula, the spreading of an infinitely narrow initial peak, $f_0(x, 0) = \delta(x)$, proceeds as follows. During a ballistic phase of propagation, that is at $t \ll 1$, the particle distribution $f_0(x, t)$ is best described by an expanding 'box' of the height $f_0 = 1/2t'$ in the region $|x| < t'$. Its edges thus propagate in opposite directions along the 'trajectories', $x = \pm t(1 - t/3)$. The thickness of the box walls, $\Delta(t)$, is much less than the size of the box, $l = 2t'$, since $\Delta/l \sim t \ll 1$. However, as the 'time' $t' = t - t^2/3$ runs slower than the real t , the box expansion slows down.

A very simple representation of the solution of Fokker-Planck equation by eq.(17) reproduces ballistic and transdiffusive propagation quite accurately. We demonstrate this by comparing it with a numerical solution of Fokker-Planck eq.(6). For further comparison, we also plot the solution of the reduced (diffusive) version of that equation, given by eq.(2). Shown in Fig.1 are all the three solutions for different times. Only at $t \sim 1$, the simplified version in eq.(17) begins to deviate from the correct solution significantly. It is worth emphasizing that the diffusive solution remains inadequate even for times close to unity. Even at $t = 2$, the simplified version of the analytic solution provides an approximation comparable with the accuracy of diffusive approximation in eq.(20). But this will, of course, be more accurate at later times and can be used instead of eq.(17), starting from $t \simeq 2$.

B. Transition to diffusive propagation

At times $t \gtrsim 1$, the simple representation of the solution in eq.(17) becomes progressively inaccurate and should be replaced by a different expansion that will time asymptotically converge to the diffusive solution. To demonstrate this convergence, we discard all terms containing powers of e^{-t} in the expansion given by eq.(12) and retain only the highest order terms for $t \gg 1$. Upon extracting such terms from each $M_{2n}(t)$ by using the solution for the moments in eqs.(9) and (14) (see also the expressions of the first few moments in Appendix), we may write the moment generation function in eq.(12) as

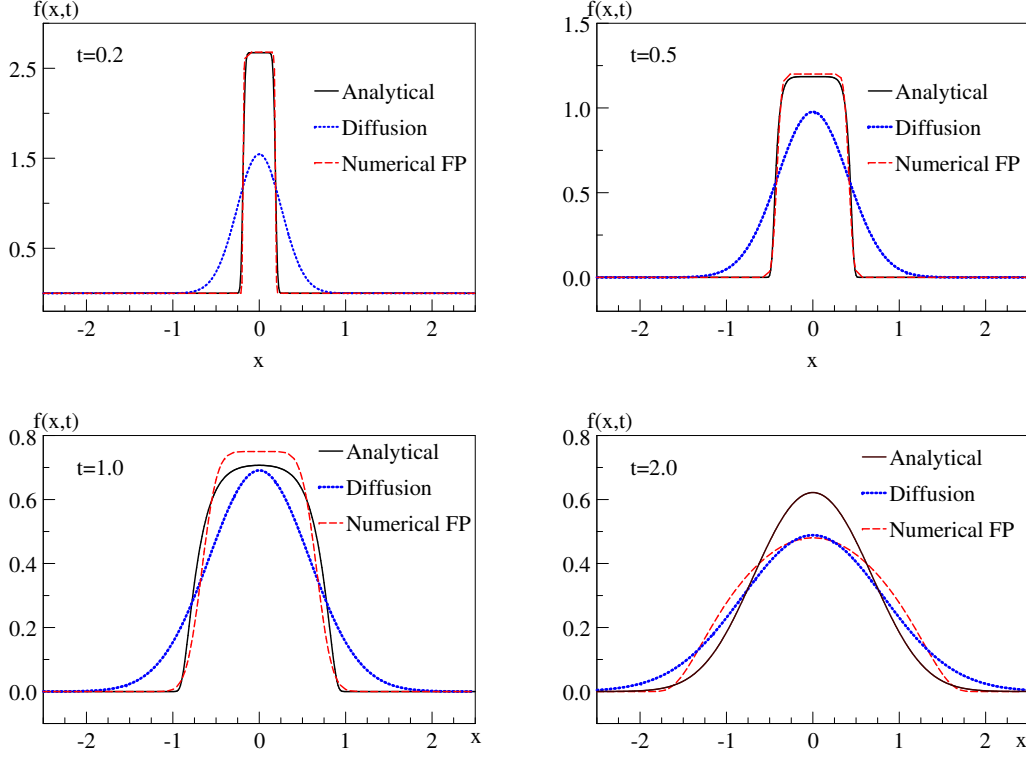


Figure 1. Fundamental solution of the Fokker-Planck equation shown for its isotropic component, $f_0(x, t) = \langle f(x, \mu, t) \rangle$ at four different times. Analytic approximation given in eq.(17) is strictly valid for $t < 1$. It becomes inadequate for $t > 3/2$ (when t' begins to decrease with t) and needs to be replaced by a different approximation, or by the diffusive solution, also shown for comparison.

$$f_\lambda(t) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{2n}(t) \approx \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!} \lambda^{2n} \left(\frac{t}{3}\right)^n \quad (18)$$

By writing $(2n-1)!!/(2n)! = 2^{-n}/n!$, the latter series can be summed up straightforwardly to yield

$$f_\lambda = e^{\lambda^2 t/6} \quad (19)$$

After replacing $\lambda = ik$ and performing an inverse Fourier transform we obtain the conventional diffusive solution

$$f_0(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx - k^2 t/6} = \sqrt{\frac{3}{2\pi t}} e^{-3x^2/2t} \quad (20)$$

Summarizing the results of this section, we have simplified the full solution of Fokker-Planck equation given by an inverse Fourier integral of a series in eq.(13). The primary focus of this derivation has been on early (ballistic and transdiffusive) phases of CR propagation ($t \lesssim 1$). By integrating the Fokker-Planck equation numerically, we have demonstrated that a simple formula in eq.(17) provides an accurate description of CR propagation including the transdiffusive phase at $t \sim 1$, in which the edges of the expanding “box” distribution broaden considerably. The derivation of a simplified formula for the transient regime between transdiffusive ($t \gtrsim 1$) and diffusive ($t \gg 1$) phases can be deferred to a later work as it is progressively well described by the simple diffusive regime represented in eq.(20).

V. CONCLUSIONS

In this paper, an exact solution of a Fokker-Planck equation given in the form of eq.(1) is obtained. The primary attention has been directed to the isotropic part of the particle distribution, $f_0(x, t)$, as it carries the information about particle propagation from a point source (fundamental solution). The evolution of f_0 has three phases, ballistic ($t < t_c$), transdiffusive ($t \sim t_c$) and diffusive ($t \gg t_c$), where t_c is the collision time. The ballistic phase is characterized by a decelerating expansion of the initial point source in form of “box” distribution with broadening walls. The next, transdiffusive phase is marked by the box walls broadened to its size and a noticeable slow down of expansion. Finally, the evolution enters the conventional diffusion phase.

No features characteristic for the solution of telegraph equation with the same initial condition (e.g., [15]) are present. These features would have appeared as two sharp peaks attached to oppositely propagating fronts of the expanding box. The absence of them in the obtained solution confirms the earlier conclusion [19] that the telegraph equation is inconsistent with its parent Fokker-Planck equation except for a late diffusive phase ($t \gg t_c$) in which there is no difference between diffusive, hyperdiffusive and telegraph equation based description of particle propagation.

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Appendix A: Higher moments of particle distribution and summation formulae

Using eq.(9), after some computer-assisted algebra, we obtain the following expressions for a few higher moments, $M_{0,2n} \equiv M_{2n}$, needed to compute the full solution given by eq.(13)

$$M_4 = \frac{1}{270}e^{-6t} - \frac{t+2}{5}e^{-2t} + \frac{1}{3}t^2 - \frac{26}{45}t + \frac{107}{270}$$

$$M_6 = \frac{1}{31500}e^{-12t} - \frac{3t+2}{1134}e^{-6t} + \frac{3}{1750}(175t^2 + 1065t + 1581)e^{-2t} + \frac{5}{9}t^3 - \frac{37}{18}t^2 + \frac{226}{63}t - \frac{6143}{2268}$$

$$M_8 = \frac{1}{6945750}e^{-20t} - \frac{5t+2}{253125}e^{-12t} + \left(\frac{t^2}{567} + \frac{11t}{11907} - \frac{59}{27783}\right)e^{-6t} - \left(\frac{14}{25}t^3 + \frac{858}{125}t^2 + \frac{151042}{5625}t + \frac{18509371}{506250}\right)e^{-2t} \\ + \frac{35}{27}t^4 - \frac{224}{27}t^3 + \frac{3554}{135}t^2 - \frac{281183}{6075}t + \frac{123403}{3375}$$

This process can be continued using the recurrence relations in eqs.(9) and (14). Our purpose, however, is to derive a simplified, easy-to-use version of the complete solution. Starting from the case $t < 1$, we first expand the moments $M_{2n}(t)$ up to the order t^{2n+2} . This gives us a series in eq.(15) for the moment-generating function which we rewrite here as follows

$$f_\lambda = 1 + \frac{\lambda^2 t^2}{3!} \left[\left(1 - \frac{t}{3}\right)^2 + \frac{2t^2}{9} \right] + \frac{\lambda^4 t^4}{5!} \left[\left(1 - \frac{t}{3}\right)^4 + \frac{28}{45}t^2 \right] + \frac{\lambda^6 t^6}{7!} \left[\left(1 - \frac{t}{3}\right)^6 + \frac{6}{5}t^2 \right] + \dots$$

By denoting $t' = t(1 - t/3)$, and separating the terms in the brackets to form two individual series, of which the first one can be summed up immediately, we obtain

$$f_\lambda = \frac{1}{2\lambda t'} \left(e^{\lambda t'} - e^{-\lambda t'} \right) + \frac{t^2}{45} S(\lambda t) \quad (\text{A1})$$

The residual series

$$S(y) \equiv \sum_{n=1}^{\infty} \frac{2n(2n+3)}{(2n+1)!} y^{2n}$$

can also be summed up by writing

$$S(y) = y \frac{d}{dy} \frac{1}{y^2} \frac{d}{dy} \sum_{n=1}^{\infty} \frac{(y)^{2n+3}}{(2n+1)!} = y \frac{d}{dy} \frac{1}{y^2} \frac{d}{dy} y^2 (\sinh y - y)$$

Using this formula, eq.(A1) rewrites

$$f_{\lambda}(t) = \frac{1}{\lambda t'} \sinh(\lambda t') + \frac{t^2}{45} \left[2 \cosh(\lambda t) + \left(\lambda t - \frac{2}{\lambda t} \right) \sinh(\lambda t) \right] \quad (\text{A2})$$

Recall, that our expansion requires $t < 1$, but λt can still be arbitrary. The leading term in eq.(A2) is the first one, while from the second (subdominant) contribution we extract the term proportional to λt . We assume it dominates ($\lambda t \gg 1$) while also being crucial in reproducing sharp features, such as propagating fronts, or CR distributions that are initially much narrower than the m.f.p. To the same order of expansion in $t < 1$, Eq.(A2) can then be written as

$$f_{\lambda}(t) \approx \frac{1}{\lambda t'} \sinh(\lambda t') \left(1 + \frac{\lambda^2 t^3 t'}{45} \right) \approx \frac{1}{\lambda t'} \sinh(\lambda t') e^{\lambda^2 t^3 t' / 45} \quad (\text{A3})$$

Because of the approximation $t' \approx t < 1$, the product of t^3 and t' in the last formula can be replaced by a different combination $t^m t'^{4-m}$, as it enters a correction term. From the standpoint of asymptotic expansion in use, there is no difference between t and t' in the highest order terms included. For example, choosing $m = 2$ instead of $m = 3$ brings some improvement of the approximation at small $t < 1$, which is used in Fig.1. The choice $m = 3$ that follows automatically from eq.(A2), makes a slightly better approximation for larger t , so we use it below and for computing the approximate solution at $t = 2$. Part of the reason for the improvement is that the modified time $t' = t - t^2/3$ needs to be expanded further when $t \sim 1$ as it decreases with t for $t > 3/2$.

Replacing λ with $-ik$ and substituting eq.(A3) into the inverse Fourier integral in eq.(13), we obtain

$$f_0(x, t) = \frac{1}{2\pi} \int e^{ikx} f_{-ik}(t) dk = \frac{1}{4\pi t'} \int_{-\infty}^x dx \int_{-\infty}^{\infty} dk (e^{ikx_-} - e^{-ikx_+}) e^{-k^2 t^3 t' / 45}$$

where $x_{\pm} = x \mp t'$. By performing a straightforward integration the result given in eq.(17) immediately follows. To conclude this brief derivation of eq.(17) we note that it ensures the conservation of the number of particles,

$$\int_{-\infty}^{\infty} f_0(x, t) dx = 1,$$

with f_0 from eq.(17), regardless of the choice of the value of m discussed above.

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